Turbulence closure problem

Modelling wind and dispersion in urban environments

Text book references: And Introduction to Boundary Meteorology, R.B. Stull, chapter 6

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Learning Objectives:

After this class, you will be able to:

- Understand the closure problem
- Basic knowledge of k-epsilon turbulence problem

Outline of the class:

1. The closure problem
2. Parameterisations
3. RANS modeling
4. Linear Eddy Viscosity approximation
5. 2-equations models
6. Wrap-up
7. One-minute notes (the muddiest point)

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The closure problem

The number of unknowns in the set of equations for turbulent flow is larger than the number of equations. The total statistical description of turbulence requires an infinite set of equations, which is called the **closure problem**

Discovered in 1924 by Keller and Friedmann, associated to the non-linear characteristics of turbulence, it has remained one of the unsolved problems of classical physics (1.000.000 $)

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3}g + f_c\varepsilon_{ij3}\overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \nu\frac{\partial^2 \overline{U_i}}{\partial x_j^2} - \boxed{\frac{\partial\left(\overline{u_i' u_j'}\right)}{\partial x_j}}$$

2nd order statistical moment

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2nd order statistical moment

Add transport equation to solve

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2nd order statistical moment

Add transport equation to solve

3rd order term appears $\frac{\partial\left(\overline{u_i' u_j' u_k'}\right)}{\partial x_j}$

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Table 6-1. Simplified example showing a tally of equations and unknowns for various statistical moments of momentum, demonstrating the closure problem for turbulent flow. The full set of equations includes even more unknowns.

Prognostic Eq. for:	Moment	Equation	Number of Eqs.	Number of Unknowns
$\overline{U_i}$	First	$\frac{\partial \overline{U_i}}{\partial t} = \ldots - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$	3	6
$\overline{u_i' u_j'}$	Second	$\frac{\partial \overline{u_i' u_j'}}{\partial t} = \ldots - \frac{\partial \overline{u_i' u_j' u_k'}}{\partial x_k}$	6	10
$\overline{u_i' u_j' u_k'}$	Third	$\frac{\partial \overline{u_i' u_j' u_k'}}{\partial t} = \ldots - \frac{\partial \overline{u_i' u_j' u_k' u_m'}}{\partial x_m}$	10	15

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Questions so far?

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Parameterisations

We make approximate the unknowns based on the known quantities —> **parameterisation**

By definition a parameterisation is an approximate to nature: we are replacing the true natural equation describing a value with some artificially constructed approximation

Why? true physics has yet to be discovered and or are too complicated for some applications

Parameterisations are rarely perfect, just adequate

There are some rules for parameterisation… out of the scope here…but anyone can develop a parameterisation respecting those rules

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Why should you care?

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RANS modelling

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3}g + f_c\varepsilon_{ij3}\overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \nu\frac{\partial^2 \overline{U_i}}{\partial x_j^2} - \boxed{\frac{\partial\left(\overline{u_i'u_j'}\right)}{\partial x_j}}$$

No change in time, 1 state

Very important how to compute the Reynolds stress?

We define the Reynolds stress based on known mean quantities:

1. **Linear eddy viscosity models**
2. Reynolds stress transport models
3. Non-linear Eddy viscosity models (Algebraic RS)
4. ….

RANS modelling

$$\cancel{\frac{\partial \overline{U_i}}{\partial t}} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3}g + f_c\varepsilon_{ij3}\overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \nu\frac{\partial^2 \overline{U_i}}{\partial x_j^2} - \boxed{\frac{\partial\left(\overline{u_i' u_j'}\right)}{\partial x_j}}$$

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Linear eddy-viscosity approximation

$$\overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3}g + f_c\varepsilon_{ij3}\overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \nu\frac{\partial^2 \overline{U_i}}{\partial x_j^2} - \boxed{\frac{\partial\left(\overline{u_i'u_j'}\right)}{\partial x_j}}$$

Boussinesq relationship:

$$-\overline{u_i'u_j'} = 2\frac{\mu_t}{\rho}S_{ij} \quad \xrightarrow{\text{where}} \quad S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

Turbulent/eddy viscosity

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3}g + f_c\varepsilon_{ij3}\overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\frac{(\mu + \boxed{\mu_t})}{\rho}\frac{\partial \overline{U_i}}{\partial x_j}\right]$$

Based on the definition of turbulent/eddy viscosity, we will have 1, 2, 3, 4 equation models

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Two-equation models

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon}$$

Turbulence kinetic energy equation
$$U_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\frac{\mu+\mu_t}{\rho\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k - \varepsilon$$

Dissipation rate equation
$$U_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\frac{\mu+\mu_t}{\rho\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon 1}\frac{\varepsilon}{k}P_k - C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

Within the model we have 5 free constants with values determined based on:

1. Decaying homogeneous isotropic turbulence $C_{2\varepsilon}$
2. Homogeneous shear flow $C_{2\varepsilon}$ $C_{1\varepsilon}$
3. Logarithmic layer $C_{2\varepsilon}$ C_μ σ_ε $C_{1\varepsilon}$
4. Experimental data…

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$\longrightarrow$ $\boxed{\text{Fixed values for our simulations}}$

Many other models…you wanna know more, ask about it!

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Wrap up:

Rest of the classes will focus on assignment 2!!! The lectures are over!

Thank you for baring me and fluid dynamics!

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