General understanding of governing equations in turbulent flow

Modelling wind and dispersion in urban environments

Text book references: And Introduction to Boundary Meteorology, R.B. Stull, chapter 3

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Learning Objectives:

After this class, you will be able to:

- Recognise the terms within the flow equations
- Apply mean averaging to the flow equations

Outline of the class:

1. Methodology

2. Basic governing equations

3. Equations for mean variables in a turbulent flow

4. Wrap-up

Outline of the class:

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Find equations terms

Outline of the class:

1. Methodology

2. Basic governing equations

Find equations terms

3. Equations for mean variables in a turbulent flow

Mean average momentum

4. Wrap-up

Outline of the class:

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Methodology

Today it's gonna be a long day….we will do some derivations, that are sometimes long, so to not lose sight of what we are doing, I'm summarising the main steps to develop *prognostic equations for mean quantities* such as temperature and wind:

Step 1	Identify the basic governing equations that apply to the ABL
Step 2	Expand the total derivatives into the local and advective contributions
Step 3	Expand dependent variables within those equations into mean and turbulent parts (perturbations)
Step 4	Apply Reynolds averaging to get the equations for mean variables within a turbulent flow
Step 5	Add the continuity equation to put the result into flux form
Step 6	Next class…
Step 7	Next class…

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Outline of the class:

1. Methodology

2. **Basic governing equations**

3. Equations for mean variables in a turbulent flow

4. Wrap-up

Basic Governing Equations

We will explore 6 basic governing equations:

1. Equation of State (Ideal Gas Law)
2. Conservation of Mass (Continuity Equation)
3. Conservation of Momentum (Newton's Second Law)
4. Conservation of Moisture
5. Conservation of Heat (First Law of Thermodynamics)
6. Conservation of a Scalar quantity (pollutant)

Basic Governing Equations

Equation of state (Ideal Gas Law)

$$p = \rho_{air} R T_v$$

p: pressure [Pa]

$\boldsymbol{\rho}_{air}$: density of moist air[kg/m^3]

R: gas constant for dry air 287K/Kkg

T_V: virtual absolute temperature

The idea gas equation describes the state of gases in the boundary layer.

We will not use it in our class, since the simulations we will run will not include phase changes, nor rain or clouds

Basic Governing Equations

Conservation of Mass (continuity equation)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \left(\rho U_j\right)}{\partial x_j} = 0$$

t: time [s]

ρ: density [kg/m3]

U_j: velocity [m/s], j=1,2,3 (U,V,W)

x_j: length space [m], j=1,2,3 (x,y,z)

$$\frac{1}{\rho}\frac{d\rho}{dt} << \frac{\partial U_j}{\partial x_j} \rightarrow$$

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If V<100m/s && L<12km && L< C_s^2/g && L< C_s/f, where Cs is the speed sound (343 m/s) and f is the frequency of any pressure waves that might occur:

$$\frac{1}{\rho}\frac{d\rho}{dt} << \frac{\partial U_j}{\partial x_j} \rightarrow$$

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"incompressible flow"

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"incompressible flow"

This equation is fundamental in CFD, whenever we solve flow around buildings, mass conservation must be ensured, this just means that whatever amount of fluid (air) enters the domain, needs to exit the domain.

It is one of the quickest check to verify that your predictions are correct!

Questions so far?

Basic Governing Equations

Conservation of Momentum (Newton's 2nd Law)

The net forces on an object is equal to the mass of that object multiplied by the acceleration of the object:

$$F = ma$$

Storage	+ Advection	=-Gravity	- Coriolis	- Pressure gradient	+Viscous stress
1	2	3	4	5	6

Basic Governing Equations

Conservation of Momentum (Newton's 2nd Law)

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$$\boldsymbol{F} = m\boldsymbol{a}$$

$$\boxed{\frac{\partial U_i}{\partial t}} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3} g - 2\varepsilon_{ijk}\Omega_j U_k - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}$$

1 2 3 4 5 6

storage

Represents the storage of momentum (inertia), which is the resistance of any object to any change in its velocity

Basic Governing Equations

Conservation of Momentum (Newton's 2nd Law)

The net forces on an object is equal to the mass of that object multiplied by the acceleration of the object:

$$\boldsymbol{F} = \boldsymbol{m}\boldsymbol{a}$$

$$\frac{\partial U_i}{\partial t} + \boxed{U_j \frac{\partial U_i}{\partial x_j}} = -\delta_{i3} g - 2\varepsilon_{ijk}\Omega_j U_k - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}$$

1 2 3 4 5 6

Advection

Describes advection, which is the transport of a substance by bulk motion

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Basic Governing Equations

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1 2 3 4 5 6

Gravity

Allows gravity to act vertically, $\boldsymbol{\delta}_{i3}$ is the "Kronecker delta"

$$\delta_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$$

Basic Governing Equations

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1 2 3 4 5 6

Coriolis

Describes the influence of the Earth's rotation (Coriolis effects), where $\mathbf{\Omega}_j$ is the angular velocity vector of the Earth's rotation:

$$\Omega_j = [0, \omega cos(\phi), \omega sin(\phi)]$$

ω: angular velocity of the Earth (2**π** radians/24h = 7.27 x 10^{-5} s^{-1})

ϕ: is the latitude

ε_{ijk}: is the alternating unit tensor

$$2\varepsilon_{ijk}\Omega_j U_k = 2\vec{\Omega} \times \vec{v} = 2\Omega\vec{v} sin(\phi)$$

$$2\varepsilon_{ijk}\Omega_j U_k = f_c \varepsilon_{ijk} U_j = 2\omega sin(\phi) \varepsilon_{ijk} U_j$$

Coriolis parameter ~10^{-4} s^{-1} (**ϕ**=44°)

Basic Governing Equations

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$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3} g - 2\varepsilon_{ijk}\Omega_j U_k \boxed{- \frac{1}{\rho}\frac{\partial p}{\partial x_i}} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}$$

1 2 3 4 5 6

Pressure gradient

Describes the pressure-gradient forces

Basic Governing Equations

Conservation of Momentum (Newton's 2nd Law)

The net forces on an object is equal to the mass of that object multiplied by the acceleration of the object:

$$\boldsymbol{F} = \boldsymbol{m}\boldsymbol{a}$$

$$\underset{1}{\frac{\partial U_i}{\partial t}} + \underset{2}{U_j \frac{\partial U_i}{\partial x_j}} = \underset{3}{-\delta_{i3} g} - \underset{4}{2\varepsilon_{ijk}\Omega_j U_k} - \underset{5}{\frac{1}{\rho}\frac{\partial p}{\partial x_i}} + \underset{6}{\boxed{\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}}}$$

Viscous stress

Represents the influence of the viscous stress

$$\frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{\mu\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right] - \left(\frac{2}{3}\right)\mu\left[\frac{\partial U_k}{\partial x_k}\right]\delta_{ij}\right\}$$

Assuming incompressibility

$$\frac{\partial \tau_{ij}}{\partial x_j} = \nu \frac{\partial^2 U_i}{\partial x_j^2}$$

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Basic Governing Equations

Conservation of Momentum (Newton's 2nd Law)

The net forces on an object is equal to the mass of that object multiplied by the acceleration of the object:

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Assuming incompressibility $\frac{\partial U_j}{\partial x_j} = 0$

$$\frac{\partial \tau_{ij}}{\partial x_j} = \nu \frac{\partial^2 U_i}{\partial x_j^2}$$

Basic Governing Equations

Conservation of Momentum (Newton's 2nd Law)

The net forces on an object is equal to the mass of that object multiplied by the acceleration of the object:

$$\boldsymbol{F = ma}$$

$$\underset{1}{\frac{\partial U_i}{\partial t}} + \underset{2}{U_j \frac{\partial U_i}{\partial x_j}} = \underset{3}{-\delta_{i3} g} \underset{4}{- 2\varepsilon_{ijk}\Omega_j U_k} \underset{5}{- \frac{1}{\rho}\frac{\partial p}{\partial x_i}} + \underset{6}{\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}}$$

$$\underset{\substack{1\\ \text{Storage}}}{\frac{\partial U_i}{\partial t}} + \underset{\substack{2\\ \text{Advection}}}{U_j \frac{\partial U_i}{\partial x_j}} = \underset{\substack{3\\ \text{Gravity}}}{-\delta_{i3} g} + \underset{\substack{4\\ \text{Coriolis}}}{f_c \varepsilon_{ij3} U_j} - \underset{\substack{5\\ \text{Pressure gradient}}}{\frac{1}{\rho}\frac{\partial p}{\partial x_i}} + \underset{\substack{6\\ \text{Viscous stress}}}{\nu \frac{\partial^2 U_i}{\partial x_j^2}}$$

Questions so far?

Outline of the class:

1. Methodology

2. Basic governing equations

3. Equations for mean variables in a turbulent flow

4. Wrap-up

Outline of the class:

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Find equations terms

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Find the equations terms!

Coriolis

Storage

$$\frac{\partial U_j}{\partial x_j} = 0$$

Continuity

$$U_j \frac{\partial U_i}{\partial x_j}$$

$$\frac{\partial U_i}{\partial t}$$

Advection

$$\delta_{i3} g$$

$$\nu \frac{\partial^2 U_i}{\partial x_j^2}$$

Pressure gradient

$$\frac{1}{\rho} \frac{\partial p}{\partial x_i}$$

Gravity

$$f_c \varepsilon_{ijk} U_j$$

Viscous stress

Mass conservation

Find the equations terms!

Coriolis

Storage

$$\frac{\partial U_j}{\partial x_j} = 0$$

Continuity

$$U_j \frac{\partial U_i}{\partial x_j}$$

$$\frac{\partial U_i}{\partial t}$$

Go to **www.menti.com**

Advect

Question **1** of **7**

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dient

Gravity

$$\rho \ \partial x_i$$

$$f_c \varepsilon_{ijk} U_j$$

Viscous stress

Mass conservation

Questions so far?

Basic Governing Equations

Conservation of Moisture

We assume q_T to be the total specific humidity of the air, which is the mass of water (all phases) per unit mass of moist air. Assuming incompressibility we can write:

$$\frac{\partial q_T}{\partial t} + U_j \frac{\partial q_T}{\partial x_j} = \nu_q \frac{\partial^2 q}{\partial x_j^2} + \frac{S_{q_T}}{\rho_{air}}$$

1	2	5	7
Storage	Advection	Diffusivity	Net moisture source term

Since $\mathbf{v}_q$ is the molecular diffusivity for water vapour in air, term 5 is analogous to viscous stress in momentum equation. Term 7 includes sources/sinks of moisture

We won't use this equation in our simulations, since we will consider dry air

Basic Governing Equations

Conservation of Heat (1st Law of Thermodynamics)

The water vapour in air not only transports sensible heat associated with its temperature, but it has the potential to release or absorb additional latent heat during any phase changes that might take place:

$$\frac{\partial \theta}{\partial t} + U_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho C_p}\left(\frac{\partial Q_j^*}{\partial x_j}\right) - \frac{L_p E}{\rho C_p}$$

1	2	5	7	8
Storage	Advection	Diffusivity	Net heat source term radiation	Net heat source term latent heat

Since $\boldsymbol{\nu_\theta}$ is the thermal diffusivity, term 5 is analogous to viscous stress in momentum equation. Term 7 and 8 includes sources/sinks of moisture

L_p is the latent heat related to the phase information E, while Q^*_j is the component of net radiation in the j^{th} direction

We won't use this equation in our simulations the year, since we will focus only on wind

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Basic Governing Equations

Conservation of scalar quantities (pollutants)

We assume C is the concentration (mass per volume [kg/m^3]) of a scalar such as CO_2 in the atmosphere. The conservation of pollutant mass follows:

$$\frac{\partial C}{\partial t} + U_j \frac{\partial C}{\partial x_j} = \nu_C \frac{\partial^2 C}{\partial x_j^2} + Sc$$

1	2	5	7
Storage	Advection	Diffusivity	Net CO_2 source term

Since $\mathbf{v}_c$ is the molecular diffusivity for CO_2 in air, term 5 is analogous to viscous stress in momentum equation. Term 7 includes sources/sinks of CO_2

Questions so far?

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Equations for Mean Variables in a Turbulent Flow

Since our problems will only focus in continuity and momentum conservation, we will only do the averaging exercise just for the continuity, as example.

Averaging rules

Reynolds averaging

Continuity equation:

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$$\overline{(A+B)} = \overline{A} + \overline{B}$$

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$$\overline{\left(\frac{dA}{dt}\right)} = \frac{d\overline{A}}{dt}$$

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Reynolds averaging

$$\overline{(Ba')} = \overline{B}\overline{a'} = \overline{B}0 = 0$$

Continuity equation:

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$$\overline{\left(\frac{dA}{dt}\right)} = \frac{d\overline{A}}{dt}$$

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$$\overline{(Ba')} = \overline{Ba'} = \overline{B}0 = 0$$

$$\overline{(A)} = \overline{(\overline{A} + \overline{a'})} = \overline{A} + \overline{a'} \rightarrow \overline{a'} = 0$$

Continuity equation:

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$$\overline{(A)} = \overline{(\overline{A} + \overline{a'})} = \overline{A} + \overline{a'} \rightarrow \overline{a'} = 0$$

$$\begin{aligned} \overline{(AB)} &= \overline{(\overline{A} + a')(\overline{B} + b')} \\ &= \overline{\overline{A}\,\overline{B} + a'\overline{B} + b'\overline{A} + a'b'} \\ &= \overline{(\overline{A}\,\overline{B})} + \overline{a'\overline{B}} + \overline{b'\overline{A}} + \overline{a'b'} \\ &= \overline{A}\,\overline{B} + \overline{a'b'} \end{aligned}$$

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Continuity equation:

$$\frac{\partial U_j}{\partial x_j} = 0$$

Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0$$

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Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0 \longrightarrow U_j = \overline{U_j} + u_j'$$

Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0 \longrightarrow U_j = \overline{U_j} + u'_j$$

$$\frac{\partial\left(\overline{U_j} + u'_j\right)}{\partial x_j} = \frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u'_j}{\partial x_j} = 0$$

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Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0 \longrightarrow U_j = \overline{U_j} + u'_j$$

Applying averaging over time:

$$\frac{\partial\left(\overline{U_j} + u'_j\right)}{\partial x_j} = \frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u'_j}{\partial x_j} = 0$$

Equations for Mean Variables in a Turbulent Flow

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Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0 \longrightarrow U_j = \overline{U_j} + u'_j$$

Applying averaging over time:

$$\frac{\partial \left(\overline{U_j} + u'_j\right)}{\partial x_j} = \frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u'_j}{\partial x_j} = 0 \longrightarrow \overline{\frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u'_j}{\partial x_j}} = 0$$

Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0 \longrightarrow U_j = \overline{U_j} + u_j'$$

Applying averaging over time:

$$\frac{\partial\left(\overline{U_j} + u_j'\right)}{\partial x_j} = \frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j} = 0 \longrightarrow \overline{\frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j}} = 0$$

$$\downarrow$$

Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0 \longrightarrow U_j = \overline{U_j} + u'_j$$

$$\frac{\partial\left(\overline{U_j} + u'_j\right)}{\partial x_j} = \frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u'_j}{\partial x_j} = 0 \xrightarrow{\text{Applying averaging over time:}} \overline{\frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u'_j}{\partial x_j}} = 0$$

$$\downarrow$$

$$\overline{\frac{\partial \overline{U_j}}{\partial x_j}} + \overline{\frac{\partial u'_j}{\partial x_j}} = 0$$

Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0 \longrightarrow U_j = \overline{U_j} + u_j'$$

$$\frac{\partial\left(\overline{U_j} + u_j'\right)}{\partial x_j} = \frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j} = 0 \xrightarrow{\text{Applying averaging over time:}} \overline{\frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j}} = 0$$

$$\downarrow$$

$$\overline{\frac{\partial \overline{U_j}}{\partial x_j}} + \cancel{\overline{\frac{\partial u_j'}{\partial x_j}}} = 0$$

Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0 \longrightarrow U_j = \overline{U_j} + u'_j$$

Applying averaging over time:

$$\frac{\partial\left(\overline{U_j} + u'_j\right)}{\partial x_j} = \frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u'_j}{\partial x_j} = 0 \longrightarrow \overline{\frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u'_j}{\partial x_j}} = 0$$

$$\downarrow$$

$$\overline{\frac{\partial \overline{U_j}}{\partial x_j}} + \cancelto{0}{\overline{\frac{\partial u'_j}{\partial x_j}}} = 0$$

Equations for Mean Variables in a Turbulent Flow

Continuity

$$\frac{\partial U_j}{\partial x_j} = 0 \longrightarrow U_j = \overline{U_j} + u'_j$$

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$$\overline{\frac{\partial \overline{U_j}}{\partial x_j}} + \underbrace{\cancel{\overline{\frac{\partial u'_j}{\partial x_j}}}}_{0} = 0$$

Equations for Mean Variables in a Turbulent Flow

Continuity

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$$\overline{\frac{\partial \overline{U_j}}{\partial x_j}} + \cancelto{0}{\overline{\frac{\partial u'_j}{\partial x_j}}} = 0$$

$$\frac{\partial \overline{U_j}}{\partial x_j} = 0$$

Questions so far?

Outline of the class:

1. Methodology

2. Basic governing equations

3. Equations for mean variables in a turbulent flow

4. Wrap-up

Outline of the class:

1. Methodology

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4. Wrap-up

Mean average momentum

Equations for Mean Variables in a Turbulent Flow

Momentum

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} U_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2}$$

?

Equations for Mean Variables in a Turbulent Flow

Momentum

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} U_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} \longrightarrow U_j = \overline{U_j} + u_j'$$

?

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Equations for Mean Variables in a Turbulent Flow

Momentum

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} U_j - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} \longrightarrow U_j = \overline{U_j} + u'_j$$

Applying averaging over time: $\longrightarrow$

?

Questions so far?

Equations for Mean Variables in a Turbulent Flow

Continuity and momentum

$$\frac{\partial \overline{U_j}}{\partial x_j} = 0$$

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3}g + f_c \varepsilon_{ij3}\overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i} + \nu \frac{\partial^2 \overline{U_i}}{\partial x_j^2} - \boxed{\frac{\partial \left(\overline{u_i' u_j'}\right)}{\partial x_j}}$$

10

Influence of Reynolds stress on the mean motions

The implication of 10: turbulence must be considered in making forecasts in turbulent flows, even if we are trying to forecast just mean quantities. The term can be large in magnitude, or larger, than many other terms in the equations.

Questions so far?

Wrap up:

1. Recognise the terms within the main flow equations

2. Apply the mean averaging to the flow equations

Next class:

General understanding of governing equations in turbulent flow

Basic Governing Equations

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} U_j - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2}$$

	2	3	4	5	6
Storage	Advection	Gravity	Coriolis	Pressure gradient	Viscous stress

Mass conservation

Continuity

Basic Governing Equations

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} U_j - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2}$$

	2	3	4	5	6
Storage	Advection	Gravity	Coriolis	Pressure gradient	Viscous stress

Mass conservation

$$\frac{\partial U_j}{\partial x_j} = 0$$

Continuity